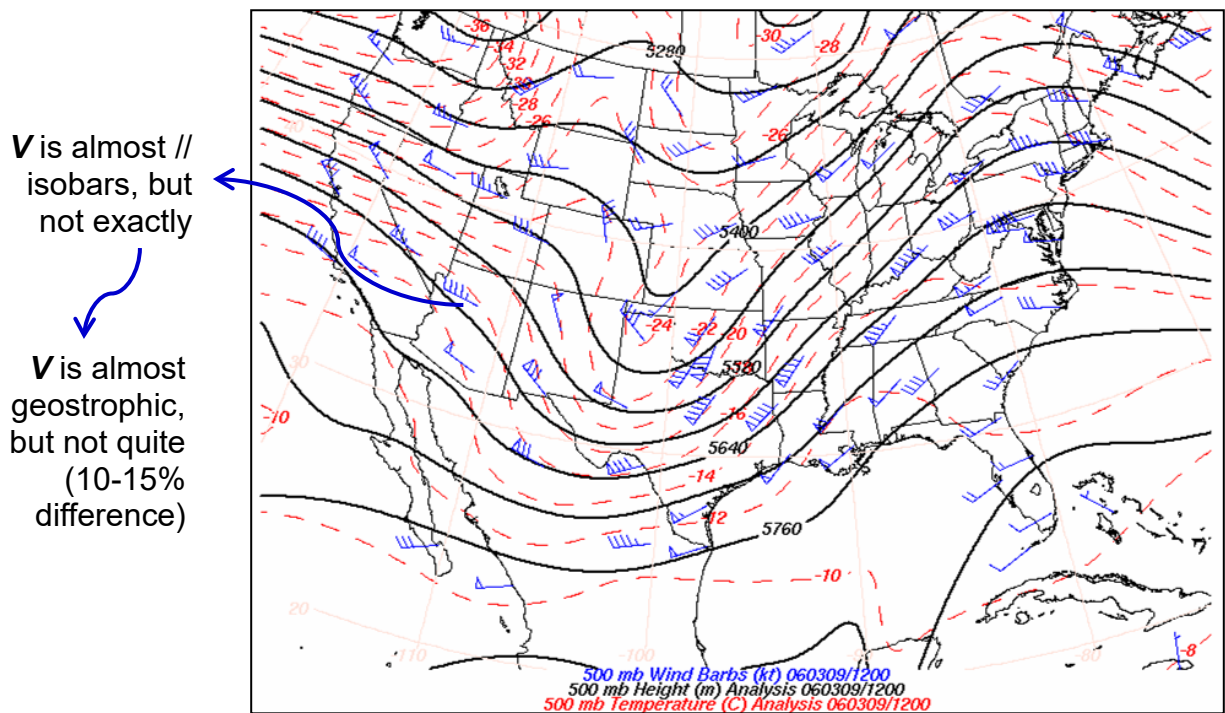


## Chapter 7 Quasi-Geostrophic (QG) Approximation and QG Vorticity Equation

### 7.1 Quasi-Geostrophic Approximation

**Purpose:** The primitive equations contain several terms of secondary significance, thus can be simplified by making the quasi-geostrophic (QG) approximation to help understand the basic dynamics of synoptic atmospheric motions in midlatitude.



The QG approximation will lead to two major equations:

- Geopotential height tendency equation for predicting the height tendency
- Omega equation for diagnosing the vertical motion

➤ QG Approximation of the momentum equations

Consider the primitive equations in isobaric coordinates

$$\frac{Du}{Dt} = fv - \frac{\partial \phi}{\partial x} \quad \text{x-momentum equation}$$

$$\frac{Dv}{Dt} = -fu - \frac{\partial \phi}{\partial y} \quad \text{y-momentum equation}$$

$$\frac{\partial \phi}{\partial p} = -\frac{RT}{p} \quad \text{Hydrostatic equation}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0 \quad \text{Continuity equation}$$

$$\frac{DT}{Dt} = \left( \frac{RT}{c_p p} \right) \omega + \left( \frac{1}{c_p} \right) \frac{DQ}{Dt} \quad \text{Thermodynamic equation}$$

➤ We have already made the following assumptions or approximations:

- (1) Inviscid
- (2) Hydrostatic
- (3) Incompressible
- (4) No Earth curvature effect

➤ Next, the total horizontal velocity is decomposed into geostrophic and ageostrophic components

$$u = u_g + u_a \quad \text{and} \quad v = v_g + v_a, \text{ or in vector form}$$

$$V = V_g + V_a$$

(5) Assume  $u_a \ll u_g$  and  $v_a \ll v_g$ , which gives

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \omega \frac{\partial u}{\partial p} \approx \frac{\partial u_g}{\partial t} + u_g \frac{\partial u_g}{\partial x} + v_g \frac{\partial u_g}{\partial y} = \frac{D_g u_g}{Dt},$$

$$\frac{Dv}{Dt} = \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \omega \frac{\partial v}{\partial p} \approx \frac{\partial v_g}{\partial t} + u_g \frac{\partial v_g}{\partial x} + v_g \frac{\partial v_g}{\partial y} = \frac{D_g v_g}{Dt},$$

$$\text{where } \frac{D_g}{Dt} = \frac{\partial}{\partial t} + u_g \frac{\partial}{\partial x} + v_g \frac{\partial}{\partial y}.$$

It is valid only when  $R_o \ll 1$ , where is the  $R_o = U/fL$  is the Rossby number.

(6) Make midlatitude  $\beta$ -plane approximation:

$$f = f_o + \frac{\partial f}{\partial y} y + \frac{1}{2!} \frac{\partial^2 f}{\partial y^2} y^2 + \dots \approx f_o + \frac{\partial f}{\partial y} y = f_o + \beta y$$

where  $\beta (= \partial f / \partial y)$  is a constant and  $y$  is the meridional distance from the latitude for  $f_o$ .

(7) Further assume  $f = f_o$  in the geostrophic wind balance equations,

$$u_g \equiv -\frac{1}{f_o} \frac{\partial \phi}{\partial y}, \quad v_g \equiv \frac{1}{f_o} \frac{\partial \phi}{\partial x}$$

With the above assumptions and approximations, the x-momentum equation becomes

$$\frac{\partial u_g}{\partial t} + u_g \frac{\partial u_g}{\partial x} + v_g \frac{\partial u_g}{\partial y} = -\frac{\partial \phi}{\partial x} + (f_o + \beta y)(v_g + v_a) \approx f_o v_a + \beta y v_g$$

$$\frac{\partial v_g}{\partial t} + u_g \frac{\partial v_g}{\partial x} + v_g \frac{\partial v_g}{\partial y} = -\frac{\partial \phi}{\partial y} - (f_0 + \beta y)(u_g + u_a) \approx -f_0 u_a - \beta y u_g$$

In summary, the QG approximation neglects the following effects:

- Friction
- Horizontal advection of momentum by the ageostrophic wind, e.g.  $\left( u_a \frac{\partial u_g}{\partial x}, v_a \frac{\partial u_g}{\partial y} \right)$ , & on ageostrophic wind, e.g.  $\left( u_a \frac{\partial u_a}{\partial x}, v_a \frac{\partial u_a}{\partial y} \right)$
- Vertical advection of momentum, e.g.  $\left( \omega \frac{\partial u_g}{\partial p}, \omega \frac{\partial u_a}{\partial p} \right)$
- Local changes in the ageostrophic wind, e.g.  $\frac{\partial u_a}{\partial t}$
- Advection of the ageostrophic momentum by the geostrophic wind, e.g.  $\left( u_g \frac{\partial u_a}{\partial x}, v_g \frac{\partial u_a}{\partial y} \right)$

### ➤ QG Continuity Equation

Substituting  $u = u_g + u_a$  and  $v = v_g + v_a$  into

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$

as earlier, assuming  $u_a$ ,  $v_a$ , and  $\omega$  are one order smaller than that of  $u_g$  and  $v_g$ , and using

$$u_g \equiv -\frac{1}{f_0} \frac{\partial \phi}{\partial y}, \quad v_g \equiv \frac{1}{f_0} \frac{\partial \phi}{\partial x}, \quad \text{and} \quad \frac{\partial u_g}{\partial x} + \frac{\partial v_g}{\partial y} = 0, \quad \text{lead to}$$

$$\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{\partial \omega}{\partial p} = 0$$

➤ QG Thermodynamic Equation

The thermodynamic energy equation in the primitive set of equations [see ASME 433 note Eq. (3.6)]

$$\frac{DT}{Dt} = \left( \frac{RT}{c_p p} \right) \omega + \left( \frac{1}{c_p} \right) \frac{DQ}{Dt}$$

can be rewritten as

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial y} - \left( \frac{\sigma p}{R} \right) \omega = \frac{1}{c_p} \frac{DQ}{Dt} \quad \text{where} \quad \sigma = - \frac{RT}{\theta p} \frac{\partial \theta}{\partial p}$$

Applying the primary QG approximation ( $u \approx u_g$  and  $v \approx v_g$ ) leads to

$$\frac{\partial T}{\partial t} = -u_g \frac{\partial T}{\partial x} - v_g \frac{\partial T}{\partial y} + \left( \frac{\sigma p}{R} \right) \omega + \frac{J}{c_p}$$

In summary, the QG equations can be written as

$$\frac{\partial u_g}{\partial t} + u_g \frac{\partial u_g}{\partial x} + v_g \frac{\partial u_g}{\partial y} = f_0 v_a + \beta y v_g \quad (6.10)$$

$$\frac{\partial v_g}{\partial t} + u_g \frac{\partial v_g}{\partial x} + v_g \frac{\partial v_g}{\partial y} = -f_0 u_a - \beta y u_g \quad (6.11)$$

$$\frac{\partial \phi}{\partial p} = - \frac{RT}{p} \quad (6.2)$$

$$\frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} + \frac{\partial \omega}{\partial p} = 0 \quad (6.12)$$

$$\frac{\partial T}{\partial t} = -u_g \frac{\partial T}{\partial x} - v_g \frac{\partial T}{\partial y} + \left( \frac{\sigma p}{R} \right) \omega + \frac{J}{c_p} \quad (6.13)$$

$$u_g = - \frac{1}{f_0} \frac{\partial \phi}{\partial y} \quad (6.7)$$

$$v_g = \frac{1}{f_0} \frac{\partial \phi}{\partial x}$$

## 7.2 Quasi-Geostrophic Vorticity Equation

➤ Start with QG equations of motion

$$\frac{\partial u_g}{\partial t} + u_g \frac{\partial u_g}{\partial x} + v_g \frac{\partial u_g}{\partial y} = f_0 v_a + \beta y v_g \quad (6.10)$$

$$\frac{\partial v_g}{\partial t} + u_g \frac{\partial v_g}{\partial x} + v_g \frac{\partial v_g}{\partial y} = -f_0 u_a - \beta y u_g \quad (6.11)$$

Taking cross differentiation of the above equations leads to the QG vorticity equation

$$\frac{\partial \zeta_g}{\partial t} + u_g \frac{\partial \zeta_g}{\partial x} + v_g \frac{\partial \zeta_g}{\partial y} = f_0 \frac{\partial \omega}{\partial p} - \beta v_g \quad (6.18)$$

where

$$\zeta_g = \frac{\partial v_g}{\partial x} - \frac{\partial u_g}{\partial y} = \frac{1}{f_0} \nabla^2 \phi \quad (6.15)$$

Equation (6.18) can be rewritten as

$$\frac{\partial \zeta_g}{\partial t} = -V_g \cdot \nabla (\zeta_g + f) + f_0 \frac{\partial \omega}{\partial p} \quad (6.18)'$$

Physical meaning of (6.18)':

LHS: local rate of change of geostrophic relative vorticity

RHS: (1) advection of absolute vorticity by geostrophic wind

(2) vorticity stretching by planetary vorticity

Substituting (6.12) into the equation leads to an alternative form of the QG vorticity equation

$$\frac{\partial \zeta_g}{\partial t} = -u_g \frac{\partial \zeta_g}{\partial x} - v_g \frac{\partial \zeta_g}{\partial y} - f_0 \left( \frac{\partial u_a}{\partial x} + \frac{\partial v_a}{\partial y} \right) - \beta v_g \quad (6.18)$$

Thus,  $\zeta_g$  can be predicted if  $\phi$  and the right-hand side of is known.